

AFFINE SURFACES WITH $AK(S) = \mathbb{C}$

T. BANDMAN AND L. MAKAR-LIMANOV

ABSTRACT. In this paper we give a description of hypersurfaces with $AK(S) = \mathbb{C}$. Let X be an affine variety and let $G(X)$ be the group generated by all \mathbb{C}^+ -actions on X . Then $AK(X) \subset \mathcal{O}(X)$ is the subring of all regular $G(X)$ -invariant functions on X . We give here a description of affine surfaces S with $AK(S) \cong \mathbb{C}$. We show that if $AK(S) = \mathbb{C}$ then S is quasihomogeneous and so may be obtained from a smooth rational projective surface by deleting a divisor of special form, which is called a “zigzag”. We denote by \mathcal{A} the set of all such surfaces, and by \mathcal{H} those which have only three components in the zigzag. We prove that for a surface S with $AK(S) \cong \mathbb{C}$ the following statements are equivalent: 1. S is isomorphic to a hypersurface; 2. S is isomorphic to a hypersurface $S' = \{(x, y, z) \in \mathbb{C}^3 | xy = p(z)\}$, where p is a polynomial with simple roots only; 3. S admits a fixed-point free \mathbb{C}^+ -action; 4. $S \in \mathcal{H}$. Moreover, if $S_1 \in \mathcal{H}$ and $S_2 \in \mathcal{A} \setminus \mathcal{H}$, then $S_1 \times \mathbb{C}^k \not\cong S_2 \times \mathbb{C}^k$ for any $k \in \mathbb{N}$.

1. Introduction.

In this paper we proceed with our research of the smooth surfaces with \mathbb{C}^+ -actions ([BML1], [BML2]). We denote by $\mathcal{O}(S)$ the ring of all regular functions on S . Let us recall that AK invariant $AK(S) \subset \mathcal{O}(S)$ of a surface S is just the subring of the ring $\mathcal{O}(S)$, consisting of those regular functions on S , which are invariant under all \mathbb{C}^+ -actions of S . This invariant can be also described as the subring of $\mathcal{O}(S)$ of all functions which are constants for all locally nilpotent derivations of $\mathcal{O}(S)$ ([KKMLR], [KML], [ML1]).

We would like to give the answer to the following question.

What are the surfaces with the trivial invariant AK ?

1991 *Mathematics Subject Classification.* Primary 13B10, 14E09; Secondary 14J26, 14J50, 14L30, 14D25, 16W50,.

Key words and phrases. Affine varieties, \mathbb{C} -actions, locally-nilpotent derivations..

The first author is supported by the Excellency Center of Academia and by the Ministry of Absorption, State of Israel, by the Emmy Nöther Institute for Mathematics of Bar-Ilan University. The second author is supported by an NSF grant.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

It is quite easy to show that the complex line \mathbb{C} is the only curve with the trivial invariant ([ML2]). It is also well known, that if $AK(S) = \mathbb{C}$ and $\mathcal{O}(S)$ is a unique factorization domain, then S is an affine complex plane \mathbb{C}^2 ([MiSu], [Su]). If we drop the UFD condition, then we have a lot of smooth surfaces with trivial invariant, e. g. any hypersurface of the form $\{xy = p(z)\} \subset \mathbb{C}^3$, where all roots of $p(z)$ are simple.

Since we did not know any other examples, we had the following working conjecture:

Conjecture. *Any smooth affine surface S with $AK(S) = \mathbb{C}$ is isomorphic to a hypersurface*

$$\{xy = p(z)\} \subset \mathbb{C}^3.$$

It turned out that this conjecture is true only with an additional assumption that S admits a fixed point free \mathbb{C}^+ -action. Also, if we assume that S is a hypersurface with $AK(S) = \mathbb{C}$, then it is indeed isomorphic to a one, defined by the equation $xy = p(z)$.

The surfaces of this kind are well-known from the 1989 because of the following remarkable fact which was discovered by Danielewski ([D]) in connection with the generalized Zariski conjecture (see also Fieseler, [Fi]). It is known that the surfaces $\{x^n y = p(z)\}$ with $n > 1$ are not isomorphic to $\{xy = p(z)\}$ (actually they are pairwise non-isomorphic). Nevertheless, the cylinders over all these surfaces are isomorphic. ($S \times \mathbb{C}^n$ is called “the cylinder over surface S ”.) So it seems natural to introduce a notion of equivalence for the surfaces, where two surfaces are equivalent when cylinders over these surfaces are isomorphic. That is why we also try to consider surfaces with $AK(S) = \mathbb{C}$ up to this equivalence. Though we are far from complete understanding, we know that there are two classes of surfaces which cannot be mixed by this equivalence relation. The first class consists of the hypersurfaces $\{xy = p(z)\}$ which were mentioned above. Here is an example of a

surface from the second class:

$$S = \left\{ (x, y, z, u) \in \mathbb{C}^4 : \begin{array}{l} xy = (z^2 - 1)z \\ zu = (y^2 - 1)y \\ xu = (z^2 - 1)(y^2 - 1) \end{array} \right\}.$$

2. Definitions and related notions.

If $AK(S) = \mathbb{C}$, then the group of automorphisms of S has a dense orbit. So it is natural to compare these surfaces with quasihomogeneous surfaces which were investigated by M. Gizatullin, V. Danilov, and J. Bertin ([G1], [G2], [GD], [Ber]).

Definition. A smooth affine surface S is called quasihomogeneous if the group $Aut(S)$ of all automorphisms of S has an orbit $U = S \setminus N$, where N is a finite set.

We will show that if $AK(S) = \mathbb{C}$, then indeed S is a quasihomogeneous surface. Therefore, S may be obtained from a smooth rational projective surface \bar{S} by deleting a divisor of special form, which is called a “zigzag” ([G1], [G2], [GD], [Ber]).

Let us denote by \mathcal{A} the set of all surfaces S with $AK(S) = \mathbb{C}$, and by \mathcal{H} those which have only three components in the zigzag.

We prove in Section 3, that a surface $S \in \mathcal{A}$ is isomorphic to a hypersurface if and only if $S \in \mathcal{H}$ (Theorem 1). In Section 4 we use this fact to prove that:

1) if $S_1 \in \mathcal{H}$ and $S_2 \in \mathcal{A} \setminus \mathcal{H}$, then the cylinders $S_1 \times \mathbb{C}^k$ and $S_2 \times \mathbb{C}^k$ cannot be isomorphic (Theorem 2);

2) a surface $S \in \mathcal{A}$ admits a fixed-point free \mathbb{C}^+ -action if and only if $S \in \mathcal{H}$ (Theorem 3).

Further on the following notations will be used:

$\mathcal{O}(X)$ – the ring of regular functions on a variety X ;

$K(S)$ – canonical divisor of a surface S ;

$[D]$ – class of linear equivalence of a divisor D ;

\tilde{D} – proper transform of a divisor D after a blow-up;

D^* – algebraic (total) transform of a divisor D after a blow-up;

(ω) , (f) – divisors of zeros of a form ω and a function f , respectively;

$Aut(S)$ – automorphism group of a surface S ;

$G(S)$ – subgroup of $Aut(S)$, generated by all C^+ -actions on a surface S ;

$OG(S)$ – a general orbit of the group $G(S)$.

\overline{A} – a Zariski closure of A if other meaning is not specified.

“General” means “belonging to a Zariski open subset”

A singular point of a rational function is a point where the function is not defined.

3. Characterization of hypersurfaces S with $AK(S) = \mathbb{C}$.

Following [Ber], [Mi1], [MiSu], by a line pencil on a surface S we mean a morphism $\rho : S \rightarrow C$ into a smooth curve C , such that the fiber $\rho^{-1}(z)$ for a general $z \in \mathbb{C}$ is isomorphic to \mathbb{C} . Then S contains a cylinder-like subset i. e. an open subset which is isomorphic to a direct product of \mathbb{C} and an open subset of C ([B], III.4). The pencils are different if their general fibers do not coincide. Any line pencil ρ over affine curve C on a surface S corresponds onto a C^+ -action φ_ρ on S , such that its general orbit coincides with a general fiber of the pencil, and corresponds to a locally nilpotent derivation (*lnd*) ∂_ρ in the ring $O(S)$ of regular functions on S , such that $\partial_\rho f = 0$ if and only if f is φ_ρ -invariant ([ML1],[KML], [S], [Mi1]). If there are two different line pencils in S , then $\rho(S) = \mathbb{C}$ (indeed, in this case $\rho(S)$ is an affine curve containing the image of a fiber of the second line pencil, and this fiber is isomorphic to \mathbb{C}). Since we are looking for the surfaces having many \mathbb{C}^+ actions, we assume further on that $C \cong \mathbb{C}$.

For a pencil ρ over \mathbb{C} one can find such a closure \overline{S} of S that the extension

$\bar{\rho} : \bar{S} \rightarrow \mathbb{P}^1$ of the map $\rho : S \rightarrow \mathbb{C}$ is regular and in the commutative diagram

$$(1) \quad \begin{array}{ccc} S & \hookrightarrow & \bar{S} \\ \rho \downarrow & & \downarrow \bar{\rho} \\ \mathbb{C} & \hookrightarrow & \mathbb{P}^1 \end{array}$$

the divisor $B = \bar{S} \setminus S$ is connected and has the following properties:

I. $B = F + D + E$, where

a) $F \cong \mathbb{P}^1$, $\bar{\rho}(F) = \mathbb{P}^1 - \mathbb{C}$,

b) $\bar{\rho}|_D : D \rightarrow \mathbb{P}^1$ is an isomorphism,

c) $E = \sum E_i + \sum H_i$, where $\bar{\rho}(H_i) \in \mathbb{C} \setminus \rho(S)$ and $\bar{\rho}(E_i) = z_i \in \rho(S)$ is a point.

Moreover, $\overline{\rho^{-1}(z_i)}$ is a union of disjoint smooth rational curves, and each of them intersects B precisely at one point.

II. B does not contain (-1) curves, except, maybe, D .

The structure of fibers is described in [Mi1], Lemma 4.4.1. If there are two different line pencils in S , then $E = \sum E_i$.

Definition. We call a closure \bar{S} a good ρ -closure of an affine surface S , if it has properties I and II.

Definition. Let $F_z = \rho^{-1}(z) = \sum_{i=1}^{i=m} n_i C_i$, where C_i are connected (and irreducible) components. If $m = 1, n_1 = 1$, then the fiber is called non-singular. The singular fiber is either non-connected or has $m = 1, n_1 > 1$.

Proposition 1. *Let S be a smooth affine surface with a line pencil ρ . Let \bar{S} be a good ρ -closure of S . Let F_{z_1}, \dots, F_{z_n} be all singular fibers of ρ , and let $F_{z_i} = \sum_{j=1}^{j=k_i} n_{i,j} C_{i,j}$ be a sum of irreducible curves $C_{i,j}$; $C_{i,j} \cong \mathbb{C}$. Then there exists a function $\alpha \in \mathcal{O}(S)$, such that*

a) α is linear along each nonsingular fiber $F_z, z \neq z_i, i = 1, \dots, n$ (i.e. $\alpha|_{F_z}$ is a non-constant linear function);

b) $\alpha|_{C_{i,j}} = \alpha_{i,j} = \text{const}$ for all $1 \leq i \leq n, 1 \leq j \leq k_i$.

Proof of Proposition 1. Let ∂_ρ be a non-zero *lnd*, corresponding to the line pencil ρ . If there is a nonsingular fiber $F_z = \rho^{-1}(z)$, such that $\partial_\rho(v)|_{F_z} = 0$ for all $v \in \mathcal{O}(S)$, then we may consider another *lnd* $\tilde{\partial}_\rho = \frac{\partial_\rho}{\rho - z}$ and repeat this procedure, if needed. Hence we may assume that ∂_ρ does not vanish identically along the nonsingular fibers of ρ .

Since ∂_ρ is a non-zero derivation, there exists a function $v \in \mathcal{O}(S)$, for which $\partial_\rho(v) \neq 0$, i.e. the minimal n for which $\partial_\rho^n(v) = 0$ is not smaller than 2. Let us take $u = \partial_\rho^{n-2}(v)$. Since $\partial_\rho^2(u) = 0$, $\partial_\rho(u) = f(z)$ depends only on $z = \rho(s), s \in S$. If $f(\tilde{z}) = 0$, $\tilde{z} \neq z_1, \dots, z_n$, then $u|_{\rho^{-1}(\tilde{z})} = u_0 = \text{const}$, and we consider a new function $\frac{u - u_0}{\rho - \tilde{z}}$.

Repeating this, we arrive at the situation, such that

- (1) $\partial_\rho u = f(z)$, where f vanishes only at the points $z_i, i = 1, \dots, n$;
- (2) u is a linear function along each fiber $\rho^{-1}(\tilde{z}), \tilde{z} \neq z_i, i = 1, \dots, n$.

We are going to show, that $u = u_i = \text{const}$ along each component $C_{i,j}$ of $F_{z_i}, i = 1, \dots, n$.

Indeed, u is linear along a general fiber, which means that the intersection $(\overline{U}_w, \overline{\rho}^{-1}(z)) = 1$ for closure \overline{U}_w , in \overline{S} of a general level curve $U_w = \{s \in S : u(s) = w\}$ and any z .

If $u|_{C_{i,j}} \neq \text{const}$, then $(\overline{U}_w, C_{i,j}) \geq 1$, and $(\overline{U}_w, \overline{\rho}^{-1}(z_i)) \geq n_{i,j}$. Thus, if $n_{i,j} > 1$, then $(\overline{U}_w, C_{i,j}) = 0$ and $u|_{C_{i,j}} = \text{const}$.

If $n_{i,j} = 1$, then the fiber is non-connected and $u|_{C_{i,j}} \neq \text{const}$ implies that U_w does not intersect $\overline{\rho}^{-1}(z_i) \setminus C_{i,j}$ for a general $w \in \mathbb{C}$. Thus, $u|_{\overline{\rho}^{-1}(z_i) \setminus C_{i,j}}$ has to be regular and constant. On the other hand u has a pole along D , so $u|_{\overline{\rho}^{-1}(z_i) \setminus C_{i,j}} = \infty$. Since u has only regular points, $u|_{C_{i,k}} = \infty$ as well, if $k \neq j$. But $u \in \mathcal{O}(S)$ so there are no components with $k \neq j$. Therefore $\rho^{-1}(z_i)$ has just one component of

multiplicity 1, which contradicts to our assumption.

Thus, we may take $\alpha = u$. \square

Proposition 2. *Any smooth affine surface S with $AK(S) \cong \mathbb{C}$ is quasihomogeneous.*

Proof of Proposition 2. Assume that ϕ and ψ are C^+ -actions on S having different orbits. Let ρ and κ be the corresponding line pencils, $\partial_\rho, \partial_\kappa$ corresponding Ind , let $R_z = \rho^{-1}(z)$ and $K_w = \kappa^{-1}(w)$ for general $z, w \in \mathbb{C}$ and let \overline{R}_z and \overline{K}_w be their closures in a good ρ -closure \overline{S} of S .

We are going to show that $S \setminus OG(S)$ is a finite set.

If a point $s \in S \setminus OG(S)$, and the fiber $R_{\rho(s)}$ is non-singular, then $R_{\rho(s)} \subset S \setminus OG(S)$ as well. Indeed, as it was shown in Proposition 1, we can choose $\partial_\rho, \partial_\kappa$ in such a way, that they do not vanish along non-singular fibers, i.e. there are no fixed points in these fibers.

For the same reason, $R_{\rho(s)}$ does not intersect a general fiber K_w , i.e. it is contained in $K_{\kappa(s)}$.

But then $\rho \neq \rho(s)$ along a general K_w , hence $\rho|_{K_w} = \text{const}$, and the fibers of these two actions coincide.

Thus, $s \in S \setminus OG(S)$ implies that $s \in R_{z_0} \cap K_{w_0}$ for singular fibers R_{z_0}, K_{w_0} .

If $S \setminus OG(S)$ is infinite, then there exists a connected component $C \subset R_{z_0} \cap K_{w_0}$ for singular fibers R_{z_0}, K_{w_0} of ρ and κ , respectively.

Let $\overline{\rho}^{-1}(z_0) = \overline{C} \cup E' \cup (\cup \overline{C}_i)$, where $E' \subset \overline{S} \setminus S$ and C_i are others components of $\rho^{-1}(z_0)$. Consider $K_w \cong \mathbb{C}$. Since the intersection $(\overline{K}_w, \overline{R}_z) \geq 1$, \overline{K}_w intersects $R_\infty = \overline{\rho}^{-1}(\infty)$. Hence, the only puncture of K_w belongs to R_∞ , and that means that $\overline{K}_w \cap E' = \emptyset$. So, κ has no singular points and has to be constant along E' . Since $E' \cap D \neq \emptyset$, $\kappa|_{E'} = \kappa|_D$ (see diagram 1 and recall that E' is connected). But $\kappa|_D = \infty$, (if it was not so, κ would be bounded and, hence, constant along a

general fiber R_z).

We got that $\kappa|_{E'} = \infty$ and has no singular points.

On the other hand κ is finite and constant along C , which implies that the point $C \cap E'$ is singular.

The contradiction shows that no such curve C exists and $S \setminus OG(S)$ is a finite set. So S is indeed quasihomogeneous. \square

Any good ρ -closure \overline{S} of S may be described by a graph $\Gamma(\overline{S})$ in the following way: the vertices of this graph are in bijection with irreducible components of the divisor $\overline{B} = \overline{S} \setminus S$, and two vertices are connected by an edge, if they intersect each other.

Now we are going to use the description of quasihomogeneous affine surfaces due to M.H. Gizatullin and J. Bertin ([G1], [G2], [GD], [Ber]).

Any such surface $S \neq \mathbb{C}^2$ may be obtained by the following blow-up process, described in [G2].

Let $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let $\overline{\rho} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a projection onto the second factor. Let $F_0 = \overline{\rho}^{-1}(z_0)$, $F_1 = \overline{\rho}^{-1}(z_1)$, $z_0, z_1 \in \mathbb{P}^1$ and let D be a section; that is, $\overline{\rho}|_D : D \rightarrow \mathbb{P}^1$ is an isomorphism.

Step 0 is an initial step, we start with the divisor, described by the following graph:

$\begin{array}{ccccc} \bullet & \text{---} & \bullet & \text{---} & \bullet \\ f & & d & & f_1 \end{array},$
 where vertices f, d, f_1 represent components F_0, D, F_1 respectively,

Step 1 is the blow-up $\sigma_1 : S_0 \rightarrow \overline{S}_1$ of a point $w_1 \in F_1$ into an exceptional component $E \subset \overline{S}_1$.

We denote $F_1^* = \tilde{F}_1 + E$ as $E_0 + E_1$, where E_0, E_1 are two rational curves, and the graph of $F_0 + D + E_1 + E_0$ looks like

$\begin{array}{cccc} \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet \\ f & & d & & e_1 & & e_0 \end{array},$

where the vertices f, d, e_1, e_0 , represent the components $\tilde{F}_0, \tilde{D}, E_1, E_0$ respectively,

Step 2 is the blow-up $\sigma_2 : \overline{S}_1 \rightarrow \overline{S}_2$ of a point $w_2 \in E_1 \cup E_0$ into a component $E_2 \subset \overline{S}_2$, in such a way that a graph of $\tilde{F}_0 + \tilde{D} + \tilde{E}_1 + \tilde{E}_0 + E_2$ is linear.

Step k is the blow-up $\sigma_k : \overline{S}_{k-1} \rightarrow \overline{S}_k$ of a point $w_k \in \tilde{E}_0 \cup \tilde{E}_1 \cup \tilde{E}_2 \cdots \cup \tilde{E}_{k-2} \cup E_{k-1}$ into a component $E_k \subset \overline{S}_k$ in such a way that the graph of the divisor $\tilde{F}_0 + \tilde{D} + \tilde{E}_0 + \tilde{E}_1 + \tilde{E}_2 + \cdots + \tilde{E}_{k-1} + E_k$ is linear.

Step k + 1 is the last step. Let $\alpha_1 \dots \alpha_q$ be different points in $\tilde{E}_0 \cup \tilde{E}_1 \cup \cdots \cup \tilde{E}_{k-1} \cup E_k$ such that each α_i belongs to one component only, $1 \leq i \leq q$. Let $\tau_1 \dots \tau_q$ be blow-ups of the points $\alpha_1 \dots \alpha_q$ into the curves G_i , $1 \leq i \leq q$, respectively; and let \overline{S} be $\tau_1 \circ \tau_2 \cdots \circ \tau_q (\overline{S}_k)$.

The desired surface $S = \overline{S} \setminus \left\{ \tilde{F}_0 \cup \tilde{D} \cup \left(\bigcup_{j=0}^k \tilde{E}_j \right) \right\}$.

Remark. *This description of quasihomogeneous surfaces implies, in particular, that there may be only one singular fiber for a line pencil ρ .*

We want to choose the “minimal” way to obtain S by the described process, i.e. to obtain a good ρ -closure of S . For this we want to substitute $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$ by a minimal ruled surface \mathbb{F}_n (see [B]).

In the sequel, for simplicity of notation we will denote \tilde{E}_j as E_j , since it cannot lead to confusion.

Proposition 3. *The surface $S \not\cong \mathbb{C}^2$, obtained by the blow-up process, described above, may be obtained by the same process, starting with the minimal surface $S_0 = \mathbb{F}_n$ and ending with such \overline{S} , that $E_j^2 \neq -1$ in \overline{S} for all $0 \leq j \leq k$.*

Proof of Proposition 3. We prove the Proposition by induction on the number of steps k . We start with the surface $S_0 = \mathbb{F}_n$ and show, that, changing n , we may always eliminate the (-1) components.

Assume that $k = 0$. Since $\rho^{-1}(z_1) \subset S$ is singular (recall that $S \not\cong \mathbb{C}^2$) there are points $\alpha_i \in F_1$, $1 \leq i \leq q$, in F_1 , which are blown up at the first (and last) step into

the curves G_i . Thus, in \bar{S} this fiber has a form $\tilde{F}_1 + \sum_{i=1}^{i=q} G_i$ (the multiplicities are equal to 1), which implies that the fiber is not connected, $q > 1$, $(\tilde{F}_1)^2 = -q < -1$.

Assume now that the Proposition is true for all $k < k_0$. Let E_j be a component of F_1^* in \bar{S}_{k_0} , such that $E_j^2 = -1$.

There are two possibilities:

1. E_j is a result of the blow up σ_j . The points of this component are not blown-up at any proceeding step, since it would make $E_j^2 < -1$. Thus, E_j may be contracted back, and we may obtain surface S by the same process, omitting the step number j , i.e. as a complement to zigzag, obtained by the blow-up process having one step less.

2. E_j is a proper transform of F_1 . In this case we may blow it down after step 1, and obtain the same surface by the same process, having one step less, starting with the surface $S_0 = \mathbb{F}_{n+1}$ or $S_0 = \mathbb{F}_{n-1}$.

By the assumption of the induction it means that the Proposition is true for k_0 . □

Definition. We denote by \mathcal{A} the class of all smooth affine surfaces S with $AK(S) = \mathbb{C}$. Let us denote by $\mathcal{H} \subset \mathcal{A}$ the subset of those surfaces, for which $k = 0$ in a good ρ -closure, obtained by the described process.

Theorem 1. *A surface $S \in \mathcal{A}$ is isomorphic to a hypersurface if and only if $S \in \mathcal{H}$.*

Proof of Theorem 1. The proof is based on a property of hypersurfaces, which was explained to the authors by V. Lin and M. Zaidenberg. We give here a proof because of lack of a reference.

Lemma 1. *Let $X \subset \mathbb{C}^n$, $n > 2$ be a smooth hypersurface. Then the canonical class $K(X)$ of X is trivial (i.e., the divisor of zeros of a holomorphic $(n-1)$ -form on X is equivalent to zero).*

Proof of Lemma 1. Let $\{z_1, \dots, z_n\}$ be coordinates in \mathbb{C}^n and let $p(z_1 \dots z_n) = 0$ be

the equation of X . Since X is irreducible and smooth, $p(z_1 \dots z_n)$ is an irreducible polynomial and the vector $\nabla p = \left(\frac{\partial p}{\partial z_1}, \dots, \frac{\partial p}{\partial z_n} \right)$ does not vanish on X .

Let $\omega = \sum_{i=1}^n (-1)^{n+i} dz_1 \wedge \dots \wedge \overset{i}{\vee} \wedge dz_n$, where $\overset{i}{\vee}$ denotes that dz_i is omitted. Let $\eta = \omega|_X$. We want to show that the divisor (η) of the form η is linearly equivalent to zero.

Choose a point $x \in X$ and a linear change of coordinates (u_1, \dots, u_n) , such that u_1, \dots, u_{n-1} are coordinates in the tangent hyperplane to X at the point x and

$$u_n(z_1, \dots, z_n) = \nabla p \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \sum_{i=1}^n \frac{\partial p}{\partial z_i}(x) \cdot z_i.$$

We denote by A the matrix

$$A = \begin{bmatrix} \frac{\partial z_1}{\partial u_1} & \dots & \frac{\partial z_n}{\partial u_1} \\ \dots & \dots & \dots \\ \frac{\partial z_1}{\partial u_n} & \dots & \frac{\partial z_n}{\partial u_n} \end{bmatrix}.$$

In local coordinates (u_1, \dots, u_{n-1}) in a neighborhood of x ,

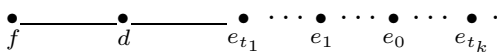
$$\eta = \sum_{i=1}^n A_{ni} du_1 \wedge \dots \wedge du_{n-1},$$

where A_{ni} are the algebraic complements of $\frac{\partial z_i}{\partial u_n}$ in the matrix A , i.e., $A_{ni} = \det A \cdot a_{in}$, where

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = A^{-1} = \begin{bmatrix} \frac{\partial u_1}{\partial z_1} & \dots & \frac{\partial u_n}{\partial z_1} \\ \dots & \dots & \dots \\ \frac{\partial u_1}{\partial z_n} & \dots & \frac{\partial u_n}{\partial z_n} \end{bmatrix}.$$

Thus, $\eta = \left(\sum_{i=1}^n \frac{\partial p}{\partial z_i} \right) \det A \cdot du_1 \wedge \dots \wedge du_{n-1}$ and the divisor $(\eta)_0$ is the zero divisor of a regular function $g(x) = \sum_{i=1}^n \frac{\partial p}{\partial z_i}$ on X . It follows that a holomorphic $(n-1)$ -form $\frac{\eta(x)}{g(x)}$ does not vanish on X , hence $[K(X)] = 0$. \square

Let $S \in \mathcal{A}$, $S \neq \mathbb{C}^2$. The graph $\Gamma(\overline{S})$ has a form:



or the form (if $k = 0$)

$$\bullet \text{---} \bullet \text{---} \bullet,$$

$f \qquad d \qquad f_1$

where the vertices f, d, f_1, e_1, e_0 , represent the components $\tilde{F}_0, \tilde{D}, \tilde{F}_1, E_1, E_0$ respectively, and a vertex e_{t_i} represents the component E_{t_i} , obtained at the step t_i .

Definition. We say that $e_i < e_j$ ($E_i < E_j$) if e_i is on the left of e_j in the graph $\Gamma(S)$.

Lemma 2. The canonical class $[K(\bar{S}_k)]$ of \bar{S}_k , $k > 0$, is the class of the divisor

$$(2) \quad K(\bar{S}_k) = \alpha \tilde{F}_0 - 2\tilde{D} - E_1 + \sum_{i=2}^k \varepsilon_i E_i,$$

where

$$(3) \quad \alpha \in \mathbb{Z}, \quad \varepsilon_i < -1 \quad \text{if} \quad e_i < e_1, \quad \text{and} \quad \varepsilon_i \geq 0 \quad \text{if} \quad e_i > e_1.$$

Let

$$F_1^k = F_1^* = \sum_{i=0}^{i=k} n_i E_i$$

be the algebraic (total) transform of F_1 in S_k . Then

$$(4) \quad \varepsilon_i < n_i - 1 \quad \text{if} \quad e_i < e_0, \quad \varepsilon_i \geq n_i \quad \text{if} \quad e_i > e_0, \quad n_1 = n_0 = 1.$$

Proof of Lemma 2.

We prove first inequalities (3) by induction on k .

The canonical class of \mathbb{F}_n is $[\alpha F_0 - 2D]$ ([B], Prop.III.18). Consider the first step: the fiber $F_1 \subset \mathbb{F}_n$ is blown up into two rational curves $F_1^* = \tilde{F}_1 + E$. Both curves have selfintersection -1 .

Two cases are possible:

$$1. \tilde{F}_1 \cap \tilde{D} = \emptyset, E \cap \tilde{D} \neq \emptyset.$$

According to the formula for the canonical class of a blow-up ([Ha], Prop. 3.3, ch. V) the canonical divisor

$$\begin{aligned} K(\overline{S}_1) &= \sigma_1^*(K(\mathbb{F}_n)) + E \\ &= \alpha\tilde{F}_0 - 2\tilde{D} - 2E + E = \alpha\tilde{F}_0 - 2\tilde{D}_0 - E. \end{aligned}$$

In this case we denote $E = E_1, \tilde{F}_1 = E_0$.

$$2. \tilde{F}_1 \cap \tilde{D} \neq \emptyset, E \cap \tilde{D} = \emptyset.$$

Then the canonical divisor

$$\begin{aligned} K(\overline{S}_1) &= \sigma_1^*(K(\mathbb{F}_n)) + E \\ &= \alpha\tilde{F}_0 - 2\tilde{D} + E = (\alpha + 1)\tilde{F}_0 - 2\tilde{D} - \tilde{F}_1. \end{aligned}$$

since $F_0 \cong E + \tilde{F}_1$. In this case we denote $E = E_0, \tilde{F}_1 = E_1$.

Thus, for $k = 1$ the formula is proved.

Assume now that (2) and (3) are proved for all $k < k_0$:

$$K(\overline{S}_{k_0-1}) = \alpha\tilde{F}_0 - 2\tilde{D} - E_1 + \sum_{i=2}^{k_0-1} \varepsilon_i E_i.$$

Then

$$\begin{aligned} K(\overline{S}_{k_0}) &= \sigma_{k_0}^*(K(\overline{S}_{k_0-1})) + E_{k_0} \\ &= \alpha\tilde{F}_0 - 2\tilde{D} - E_1 + \sum_{i=2}^{k_0-1} \varepsilon_i \cdot E_i + \varepsilon_{k_0} E_{k_0}. \end{aligned}$$

Consider the following cases:

I. At the step k_0 we blow up a point w_{k_0} , belonging only to the component E_s , represented by the vertex on the far right ($e_s \geq e_r$ for all $r < k_0$). In this case,

e_s is on the right of e_1 . By the induction assumption, we have $\varepsilon_s \geq 0$, and $\varepsilon_{k_0} = (\varepsilon_s + 1) > 0$.

II. At the step k_0 we blow up the meeting point $E_s \cap E_{s'}$, where $e_s < e_{s'} \leq e_1$. Then $\varepsilon_s < -1$, $\varepsilon_{s'} \leq -1$, $\varepsilon_{k_0} = \varepsilon_s + \varepsilon_{s'} + 1 < -1 - 1 + 1 < -1$.

III. At the step k_0 we blow up the meeting point $E_s \cap E_{s'}$, where $e_s > e_{s'} \geq e_1$. Then $\varepsilon_s \geq 0$, $\varepsilon_{s'} \geq -1$, $\varepsilon_{k_0} = \varepsilon_s + \varepsilon_{s'} + 1 \geq -1 + 1 \geq 0$.

IV. At the step number k_0 we blow up the meeting point $E_s \cap \tilde{D}$. Then $e_s \leq e_1$, $\varepsilon_{k_0} = \varepsilon_s - 2 + 1 \leq -1 - 1 < -1$.

Since the graph $\Gamma(S)$ is linear, we exhausted all the possibilities.

Now let us prove the inequalities (4).

For $k = 1$ we have $F_1^1 = E_1 + E_0$, $K(\overline{S}_1) = \alpha \tilde{F}_0 - 2\tilde{D} - E_1$, therefore, $\varepsilon_1 < n_1 - 1$.

We prove (4) for any k by induction. Assume that it is proved for all $k < k_0$. Then in \overline{S}_{k_0} we have

$$F_1^{k_0} = \sigma_{k_0}^*(F_1^{k_0-1}) = \sum_{i=0}^{i=k_0-1} n_i E_i + n_{k_0} E_{k_0},$$

where $n_{k_0} = n_s + n_r$, if E_{k_0} appears as a blow-up of the intersection $E_s \cap E_r$, or $n_{k_0} = n_s$ if E_{k_0} is the result of a blow-up of either $D \cap E_s$ or of a point of the maximal component.

Using the inequalities (4) for $k < k_0$, we get:

$$n_{k_0} = n_s \leq \varepsilon_s < \varepsilon_s + 1 = \varepsilon_{k_0}, \text{ if } E_s \text{ is the maximal component and } s \neq 0;$$

$$n_{k_0} = n_0 = 1 \leq 1 = \varepsilon_{k_0}, \text{ if } E_0 \text{ is the maximal component};$$

$$n_{k_0} = n_s + n_r \leq \varepsilon_s + \varepsilon_r < \varepsilon_s + \varepsilon_r + 1 = \varepsilon_{k_0}, \text{ if } e_0 < e_s < e_r;$$

$$n_{k_0} = n_0 + n_r = 1 + n_r \leq 0 + \varepsilon_r + 1 = \varepsilon_{k_0}, \text{ if } e_0 = e_s < e_r;$$

$$n_{k_0} = n_s + n_0 = 1 + n_s > 1 + \varepsilon_s + 1 = \varepsilon_{k_0} + 1, \text{ if } e_s < e_r = e_0;$$

$$n_{k_0} = n_s + n_r > \varepsilon_s + 1 + \varepsilon_r + 1 = \varepsilon_{k_0} + 1, \text{ if } e_s < e_r < e_0;$$

$$n_{k_0} = n_s > \varepsilon_s + 1 = \varepsilon_{k_0} + 2 > \varepsilon_{k_0} + 1, \text{ if } E_s \text{ is the minimal component.} \quad \square$$

Lemma 3. Denote the transform of F_1 in \overline{S}

$$F_1^{k+1} = F_1^* = \sum_{i=0}^{i=k} n_i E_i + \sum_{i=1}^{i=q} g_i G_i,$$

where $n_1 = n_0 = 1, g_i > 0, n_i > 0$.

$[K(S)] = 0$ if and only if the divisor $K(\overline{S})$ is equivalent to a linear combination

$$(5) \quad \sum_{i=0}^{i=k} \alpha_i E_i + f \tilde{F}_0 + d \tilde{D} + m \sum_{i=1}^{i=q} g_i G_i,$$

for some $m \in \mathbb{Z}$.

Proof of Lemma 3.

$$(6) \quad \begin{aligned} K(\overline{S}) &= K(\overline{S}_k)^* + \sum G_i \\ &= \alpha \tilde{F}_0 - 2\tilde{D} - E_1 + \sum_{i=1}^k \varepsilon_i E_i + \sum_{i=1}^q \delta_i G_i, \end{aligned}$$

where $\delta_i = \varepsilon_s + 1$ for each G_i intersecting E_s .

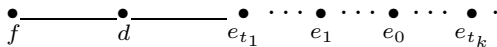
If $[K(S)] = 0$, then $K(S)$ is divisor of a rational function h , which has zeros and poles in S only along components G_i . But then h does not vanish and has no poles in any fiber F_z , $z \neq z_1$. Since general fiber is isomorphic to \mathbb{C} , it means, that h is constant along each fiber, i.e. $h(s) = (\rho(s) - z_1)^m$. But then $\delta_i = mg_i$. \square

Definition. We call component E_s essential, if there is a component G_{i_s} of the fiber $F_1^* \subset \overline{S}$ such that $G_{i_s} \cap E_s \neq \emptyset$.

Remark. We see from the previous Lemma that $[K(S)] = 0$ implies $\varepsilon_s + 1 = mn_s$ for any essential component E_s .

Lemma 4. If $k > 0$, then $[K(S)] \neq 0$.

Proof of Lemma 4. Consider the graph



Assume that $[K(S)] = 0$, i.e. $\varepsilon_s + 1 = mn_s$. Several cases, concerning the place of essential components in the graph, are possible.

I. There is an essential component E_s such that $e_s \geq e_0$. Then according to Lemma 2 $n_s \leq \varepsilon_s + 1 = mn_s$ and so $m \geq 1$.

II. There is an essential component E_s such that $e_1 < e_s < e_0$. Then according to Lemma 2 $n_s > \varepsilon_s + 1 = mn_s > 0$ and hence $1 > m > 0$.

III. There is an essential component E_s such that $e_s \leq e_1$. Then according to Lemma 2 $0 \geq \varepsilon_s + 1 = mn_s$ and $m \leq 0$.

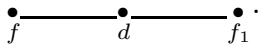
Therefore we may have only one of these cases for all essential components.

Let us assume that $e_s \leq e_1$ for any essential component E_s . Let $t_0 = \max\{t : e_t > e_1, t \geq 0\}$. By the construction $(E_t)^2 = -1$ in \overline{S}_k , (it is the result of a blow-up). So it should contain a point which is blown up at the last $k+1$ step. But then E_t is essential, which is impossible in this case.

The case $e_s \geq e_0$ for all essential components can be treated analogously since the last component to the left of E_0 also must be essential.

The remaining case II is impossible since $m \in \mathbb{Z}$.

Therefore (5) can be true only if the graph has three components:



□

Lemma 5. *If $k = 0$, then S is a hypersurface.*

Proof of Lemma 5. Let $\rho : S \rightarrow \mathbb{C}$ be a line pencil in S , $\overline{\rho}$ its extension to a good ρ -closure \overline{S} of S , φ_ρ and ∂_ρ the corresponding \mathbb{C}^+ -action and lnd respectively. Let $\rho^{-1}(0)$ be the only singular fiber. All the multiplicities are 1 in this case, so the fiber cannot be connected. Let $u \in O(S)$ be a function such that

- (1) $\partial_\rho u = \rho^n$;
- (2) u is a linear function along each fiber $\rho^{-1}(z)$, $z \neq 0$;
- (3) $u = u_i = \text{const}$ along each component G_i of $\rho^{-1}(0)$, $i = 1, \dots, q$.

Such a function exists due to Proposition 1. We are going to show that we can choose u , so that $u_i \neq u_j$, when $i \neq j$, and the rational extension \bar{u} of u to \bar{S} is finite and non-constant along \tilde{F}_1 . Indeed, u is linear along a general fiber, which means that the intersection $(\bar{U}_w, \bar{F}_z) = 1$ for the closures of a general level curve $U_w = \{s \in S : u(s) = w\}$ and the closure \bar{F}_z of a general fiber $F_z = \{s \in S : \rho(s) = z\}$.

There are three possibilities:

I. $\bar{u}|_{\tilde{F}_1} = u_0 \in \mathbb{C}$, and $u_0 \neq u_1 = \bar{u}|_{G_1}$. Then the intersection $G_1 \cap \tilde{F}_1 = \alpha_1$ is a singular point, and a general level curve passes through α_1 . Another singular point $\alpha_2 = D \cap \tilde{F}_1$, since $\bar{u}|_D = \infty$. Thus, a general level curve U_w has to pass through α_2 as well. But this contradicts to $(\bar{U}_w, \bar{F}_z) = 1$.

Thus, $\bar{u}|_{\tilde{F}_1} = u_0 \in \mathbb{C}$ implies $u_0 = u_1 = u_2 = \dots = u_q$ and we can consider a new function $\frac{u-u_0}{\rho}$ instead of u (because $F_1^* = \tilde{F}_1 + \sum G_i$, i.e. ρ has a simple zero along each component).

II. \bar{u} has a pole along \tilde{F}_1 . Then each point $\alpha_i = \tilde{F}_1 \cap G_i$, $i = 1, \dots, q$, should be a singular point of \bar{u} , and \bar{U}_w should pass through each α_i . From $(\bar{U}_w, \bar{F}_z) = 1$, it follows that there is only one component G_1 and the fiber $\rho^{-1}(0)$ is connected in this case.

Then $S \simeq \mathbb{C}^2$ (see, for example, [Su]), and is evidently isomorphic to a hyper-surface.

III. \bar{u} is not constant along \tilde{F}_1 . Because $(\bar{U}_w, \tilde{F}_1) = 1$ for a general w , it takes every value only once along \tilde{F}_1 . From $G_i \cap G_j = \emptyset$, it follows that $u_i \neq u_j$ for $i \neq j$, $i, j = 1, \dots, s$.

Consider a polynomial $p(u) = (u - u_1) \dots (u - u_q)$ and $\bar{v} = \frac{p(\bar{u})}{\rho}$.

Since \bar{u} is finite along \tilde{F}_1 , \bar{v} is regular and finite at all the points of S and it has a simple pole along \tilde{F}_1 .

Let $A_j = H_j + G_j$ be the divisor $u = u_j$. Since $(\bar{U}_w, \tilde{F}_1) = 1$ for a general w ,

$(A_j, \tilde{F}_1) = 1$ and $(H_j, \tilde{F}_1) = (A_j, \tilde{F}_1) - (G_j, \tilde{F}_1) = 0$. Thus, \tilde{F}_1 does not intersect zeros of function \bar{v} . In particular, the intersection points $s_j = G_i \cap \tilde{F}_1$ are not singular for \bar{v} , restriction $\bar{v}|_{G_i}$ has simple poles in s_j and is linear along each G_i , $i = 1, \dots, q$ (i.e., takes every value $z \in \mathbb{P}^1$ precisely at one point of G_i).

The restriction of \bar{v} on S we denote by v , $v \in O(S)$.

We define a regular map $\phi : S \rightarrow \mathbb{C}^3$ as $\phi(s) = (\rho(s), v(s), u(s))$, and we want to show that ϕ is an isomorphism of S onto a hypersurface

$$S' = \{(x, y, t) \in \mathbb{C}^3 | xy = p(t)\} \subset \mathbb{C}^3.$$

A) ϕ is an embedding. Indeed, the functions ρ and u divide points in $(S \setminus (\cup G_i))$, since ρ divides fibers of a line pencil, and u is linear along each fiber $\rho^{-1}(z)$, $z \neq 0$.

The values $u|_{G_i} = u_i$ provide the distinction between the components G_i of $\rho^{-1}(0)$, since $u_i \neq u_j$ when $i \neq j$.

The function v is linear along each G_i , so its values are different in the different points of each G_i .

B) ϕ is onto. Let $s' = S'$ and $s' = (x', y', t')$. If $x' \neq 0$, then in the fiber $\rho^{-1}(x')$ there is a point, such that $u(s) = t'$. (Indeed, $\rho^{-1}(x') \cong \mathbb{C}$ and $u|_{\rho^{-1}(x')}$ is linear.) Now, $v(s) = \frac{p(u)}{\rho} = \frac{p(t')}{x'} = y'$, so $\phi(s) = s'$.

If $x' = 0$, then $p(t') = 0$, so $t = u_j$ for some $1 \leq j \leq q$. The function v is linear along the component G_j , so there is a point $s \in G_j$ such that $v(s) = y'$. Then $\phi(s) = (0, y', u_j) = (0, y', t') = s' \in S'$. \square

We proceed with the proof of Theorem 1. Any surface $S \in \mathcal{H}$ is a hypersurface by Lemma 5. If $S \in \mathcal{A}$ but $S \notin \mathcal{H}$, then, by Lemma 4, $[K(S)] \neq 0$, and, by Lemma 1, it cannot be isomorphic to a hypersurface. \square

An example of a surface $S \in \mathcal{A} \setminus \mathcal{H}$ was given in the Introduction: $S \subset \mathbb{C}^4$ is defined by equations

$$\begin{cases} xy = (z^2 - 1)z \\ zu = (y^2 - 1)y \\ xu = (y^2 - 1)(z^2 - 1) \end{cases}$$

We will show below that this surface is not isomorphic to a hypersurface. On the other hand, there are two locally nilpotent derivations (lnd) defined in the ring $O(S)$, namely:

$$\begin{cases} \partial_1 x = 0 \\ \partial_1 z = x^2 \\ \partial_1 y = (3z^2 - 1)x \\ \partial_1 u = 2z(y^2 - 1)x + 2y(z^2 - 1)(3z^2 - 1) \end{cases}$$

$$\begin{cases} \partial_2 u = 0 \\ \partial_2 y = u^2 \\ \partial_2 z = (3y^2 - 1)u \\ \partial_2 x = 2y(z^2 - 1)u + 2z(y^2 - 1)(3y^2 - 1) \end{cases}$$

It follows that $AK(S) = \mathbb{C}$.

Corollary of Lemma 1. *The surface $S \subset \mathbb{C}^4$ defined by equations*

$$\begin{cases} xy = (z^2 - 1)z \\ zu = (y^2 - 1)y \\ xu = (y^2 - 1)(z^2 - 1) \end{cases}$$

is not isomorphic to a hypersurface.

Proof. Consider the 2-form $w = \frac{dx \wedge dz}{x}$. It is regular in Zariski open subset $U_0 = \{(x, y, z, u) \in S \mid x \neq 0\}$, where (x, z) are the local coordinates.

The fiber $\{x = 0\}$ consists of 4 components:

$$\begin{aligned} G_1 &= \{x = 0, z = 1\}; & G_2 &= \{x = 0, z = -1\}; \\ G_3 &= \{x = 0, z = 0, y = 1\}; & G_4 &= \{x = 0, z = 0, y = -1\}. \end{aligned}$$

We consider Zariski open neighborhoods U_1, U_2, U_3, U_4 , respectively, of these components:

$$U_1 = \{(x, y, z, u) \in S \mid z \neq 0, z \neq -1\} \text{ with local coordinates } \varphi_1 = \frac{z-1}{x},$$

$$\psi_1 = x;$$

$U_2 = \{(x, y, z, u) \in S \mid z \neq 0, z \neq 1\}$ with local coordinates $\varphi_2 = \frac{z+1}{x}$, $\psi_2 = x$;

$U_3 = \{(x, y, z, u) \in S \mid z^2 \neq 1, y \neq 0, y \neq -1\}$ with local coordinates $\varphi_3 = \frac{y-1}{z}$, $\psi_3 = z$;

$U_4 = \{(x, y, z, u) \in S \mid z^2 \neq 1, y \neq 0, y \neq 1\}$ with local coordinates $\varphi_4 = \frac{y+1}{z}$, $\psi_4 = z$.

Rewriting ω in these coordinates, we obtain:

$$\omega = \frac{dx \wedge dz}{x} \quad \text{in } U_0$$

$$\omega = d\psi_1 \wedge d\varphi_1 \quad \text{in } U_1$$

$$\omega = d\psi_2 \wedge d\varphi_2 \quad \text{in } U_2$$

$$\omega = -\frac{\psi_3 d\varphi_3 \wedge d\psi_3}{\varphi_3 \psi_3 + 1} \quad \text{in } U_3$$

$$\omega = -\frac{\psi_4 d\varphi_4 \wedge d\psi_4}{\varphi_4 \psi_4 - 1} \quad \text{in } U_4$$

Since $\varphi_3 \psi_3 + 1 = y \neq 0$ in U_3 and $\varphi_4 \psi_4 - 1 = y \neq 0$ in U_4 , this form is holomorphic everywhere on S .

But $\omega|_{G_3} = \omega|_{G_4} = 0$ and the divisor $(\omega) = G_3 + G_4$ is not equivalent to zero on S by Lemma 3. So, by Lemma 2, the surface S cannot be isomorphic to a hypersurface. \square

4. Corollaries for cylinders and \mathbb{C}^+ -actions.

Theorem 2. *Let S_1, S_2 be smooth affine surfaces, such that $S_1 \in \mathcal{H}$ and $S_2 \in \mathcal{A} \setminus \mathcal{H}$. Then $S_1 \times \mathbb{C}^k \not\simeq S_2 \times \mathbb{C}^k$ for any $k \in \mathbb{N}$.*

Proof of Theorem 2. Assume, to the contrary, that $S_1 \times \mathbb{C}^k \simeq S_2 \times \mathbb{C}^k = W$. Since $S_1 \in \mathcal{H}$, by Theorem 1, it is isomorphic to a hypersurface $S \subset \mathbb{C}^3$, and $W \simeq S \times \mathbb{C}^k$ is a hypersurface in \mathbb{C}^{k+3} as well. By Lemma 1, for any holomorphic $(k+2)$ -form ω on W , the divisor (ω) has to be linearly equivalent to zero, i.e., there should be a rational function f_ω such that $(\omega) = (f_\omega)$.

Take any holomorphic 2-form ω' on S_2 . Let z_1, \dots, z_k be some global coordinates in \mathbb{C}^k . Then the form $\omega = \omega' \wedge dz_1 \cdots \wedge dz_k$ is a holomorphic $k+2$ -form on W . Hence, $(\omega) = (f_\omega)$ for a function f_ω . But then $[(\omega')] = [(f_\omega)|_{S_2}] = 0$. It follows that $[K(S_2)] = 0$ and, due to Theorem 1, $S_2 \in \mathcal{H}$. \square

Theorem 3. *A surface $S \in \mathcal{A}$ admits a fixed-point \mathbb{C}^+ -action if and only if $S \in \mathcal{H}$.*

Proof of Theorem 3. Let $S \in \mathcal{A}$ and let φ_ρ be a fixed-point free \mathbb{C}^+ -action, let ρ be a corresponding line pencil and let $\rho^{-1}(0)$ consist of q components G_1, \dots, G_q . Consider another surface $S_q = \{xy = (z-1)\cdots(z-q)\} \subset \mathbb{C}^3$. This surface is smooth, affine and has two \mathbb{C}^+ -actions; namely,

$$\varphi_x^\lambda(x, y, z) = (x, \frac{(z + \lambda x - 1) \cdots (z + \lambda x - q)}{x}, z + \lambda x)$$

and

$$\varphi_y^\lambda(x, y, z) = (\frac{(z + \lambda y - 1) \cdots (z + \lambda y - q)}{y}, y, z + \lambda y).$$

Thus, $S_q \in \mathcal{A}$. The actions φ_x^λ and φ_y^λ have no fixed points, because the corresponding $\ln d$'s, namely:

$$\partial_x : \partial_x(x) = 0, \partial_x(z) = x, \partial_x(y) = p'(z);$$

$$\partial_y : \partial_y(y) = 0, \partial_y(z) = y, \partial_y(x) = p'(z);$$

never vanish.

The fibers of the line pencil ρ_x in S_q corresponding to ∂_x are the curves $\{x = \text{const}\}$. All of them are connected except the fiber $x = 0$, which has q connected components.

The fibers of the line pencil ρ in S have precisely the same structure.

By the Theorem of Daniliewski and Fieseler ([D], [Fi]) the cylinders $S \times \mathbb{C} \simeq S_q \times \mathbb{C}$.

But S_q is a hypersurface, hence $S_q \in \mathcal{H}$ by Theorem 1. Due to Theorem 2, $S \in \mathcal{H}$ as well.

Therefore, if S admits a fixed-point-free \mathbb{C}^+ -action, then $S \in \mathcal{H}$.

Now, assume that $S \in \mathcal{H}$. As it was shown in Lemma 5, it is isomorphic to the surface

$$S' = \{(x, y, z) \in \mathbb{C}^3 | xy = p(t)\} \subset \mathbb{C}^3.$$

Since S is smooth, all the roots t_1, \dots, t_q of $p(t)$ are simple. That is why the $\text{ind } \partial$, defined as

$$\partial : \partial(x) = 0, \partial(t) = x, \partial(y) = p'(t);$$

does not vanish on S' . But then the \mathbb{C}^+ -action, defined by ∂ has no fixed points.

□

Acknowledgments. We thank V. Lin and M. Zaidenberg for the idea of the proof of Lemma 1. It is our pleasure to thank M. Gizatullin for the discussions concerning quasihomogeneous surfaces, and M. Miyanishi and R.V. Gurjar for the most helpful discussions and examples.

REFERENCES

- [B] A. Beauville, *Complex algebraic surfaces* London Math. Soc. Lecture notes, vol. 66, 1983.
- [BML1] T. Bandman, L. Makar-Limanov, *Cylinders over affine surfaces*, Japan. Jour. Math. **26** (2000), 208-217.
- [BML2] T. Bandman, L. Makar-Limanov, *Affine Surfaces with isomorphic cylinders*, preprint, e-prints 0001067..
- [Ber] J. Bertin, *Pinceaux de droites et automorphismes des surfaces affines*, J. Reine und Angew. Math. **341** (1983), 32-53.
- [D] W. Danielewski, *On the cancellation problem and automorphism groups of affine algebraic varieties*, preprint, Warsaw (1989).
- [Fi] K.-H. Fieseler, *On complex affine surfaces with \mathbb{C}^+ -action*, Comment. Math. Helvetici **69** (1994), 5-27.
- [G1] M.H. Gizatullin, *Invariants of incomplete algebraic surfaces obtained by completions*, Math. USSR Izvestiya **5** (1971), 503-516.
- [G2] M.H. Gizatullin, *Quasihomogeneous affine surfaces*, Math. USSR Izvestiya **5** (1971), 1057-1081.
- [GD] M.H. Gizatullin, V.I. Danilov, *Automorphisms of affine surfaces. I*, Math. USSR Izvestiya **9** (1975), 493-534.

- [Ha] Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, Berlin, 1977.
- [KKMLR] S. Kaliman, M. Koras, L. Makar-Limanov, P. Russell, *C^* -actions on C^3 are linearizable*, ERA-AMS **3** (1997), 63-71.
- [KML] S. Kaliman, L. Makar-Limanov, *On the Russell-Koras contractible threefolds*, Journ. of the Algebraic Geometry **6(2)** (1997), 247-268.
- [ML1] L. Makar-Limanov, *Locally nilpotent derivations, a new ring invariant and applications*, preprint.
- [ML2] L. Makar-Limanov, *Cancellation for curves*, preprint.
- [Mi1] M. Miyanishi, *Non-complete algebraic surfaces*, *Lecture Notes in Math.*, vol. 857, Springer-Verlag, 1981.
- [MiSu] M. Miyanishi, T. Sugie, *Affine surfaces containing cylinderlike open set*, J. Math. Univ. Kyoto **20** (1980), 11-42.
- [S] M. Snow, *Unipotent actions on affine space*, Topological methods in Algebraic Transformation groups Progress in Math., vol. 80, 1989, pp. 165-177.
- [Su] T. Sugie, *Algebraic characterization of the affine plane and the affine 3-space*, Topological methods in Algebraic Transformation groups Progress in Math., vol. 80, 1989, pp. 177-190.

Tatiana M. Bandman, Dept. of Mathematics & CS, Bar-Ilan University, Ramat-Gan, 52900, Israel, e-mail: bandman@macs.biu.ac.il.

Leonid Makar-Limanov, Dept. of Mathematics & CS, Bar-Ilan University, Ramat-Gan, 52900, Israel, e-mail: lml@macs.biu.ac.il; Dept. of Mathematics, Wayne State University, Detroit, MI 48202, USA, e-mail: lml@math.wayne.edu.